UNIVERSITY OF WATERLOO
FACULTY OF ENGINEERING
Department of Electrical &
Computer Engineering

ECE 204 *Numerical methods*

Approximating a point using least-squares best-fitting polynomials

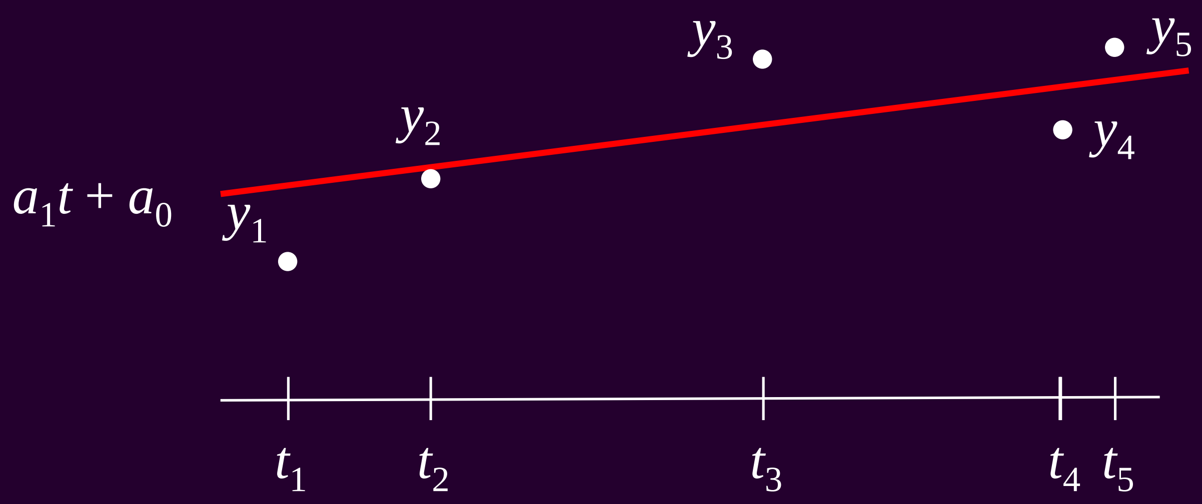
Douglas Wilhelm Harder, LEL, M.Math.
dwharder@uwaterloo.ca
dwharder@gmail.com

Introduction

- In this topic, we will
  - Discuss evaluating a least-squares best-fitting polynomial at a point
  - Describe how to find the coefficients of that polynomial
  - Look at the change in run time
    - We'll reduce the run time to O(1)!
  - Observing the differences between linear and quadratic interpolating polynomials

Review

- From the main discussion:
  - Suppose we have found a least-squares best-fitting linear polynomial passing through a set of given noisy points
    - We can thus evaluate the linear polynomial at any point on the line

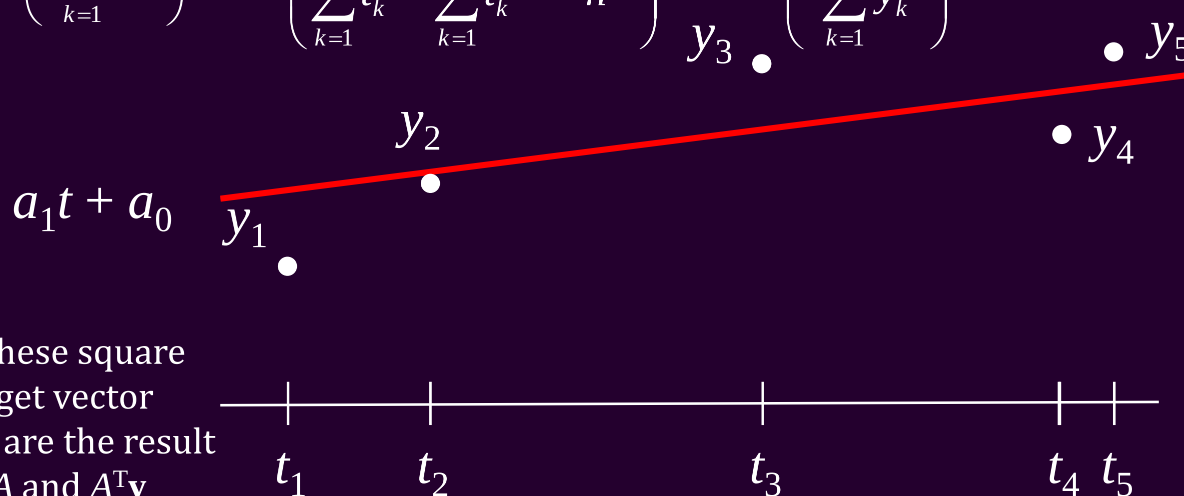


Review

- Problem:
  - Finding the least-squares best-fitting polynomial requires first calculating and then solving these systems of linear equations

$$\begin{pmatrix} \sum_{k=1}^{n} t_k^2 & \sum_{k=1}^{n} t_k \\ \sum_{k=1}^{n} t_k & n \end{pmatrix} \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{n} t_k y_k \\ \sum_{k=1}^{n} y_k \end{pmatrix}$$

$$\begin{pmatrix} \sum_{k=1}^{n} t_k^4 & \sum_{k=1}^{n} t_k^3 & \sum_{k=1}^{n} t_k^2 \\ \sum_{k=1}^{n} t_k^3 & \sum_{k=1}^{n} t_k^2 & \sum_{k=1}^{n} t_k \\ \sum_{k=1}^{n} t_k^2 & \sum_{k=1}^{n} t_k & n \end{pmatrix} \begin{pmatrix} a_2 \\ a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{n} t_k^2 y_k \\ \sum_{k=1}^{n} t_k y_k \\ \sum_{k=1}^{n} y_k \end{pmatrix}$$

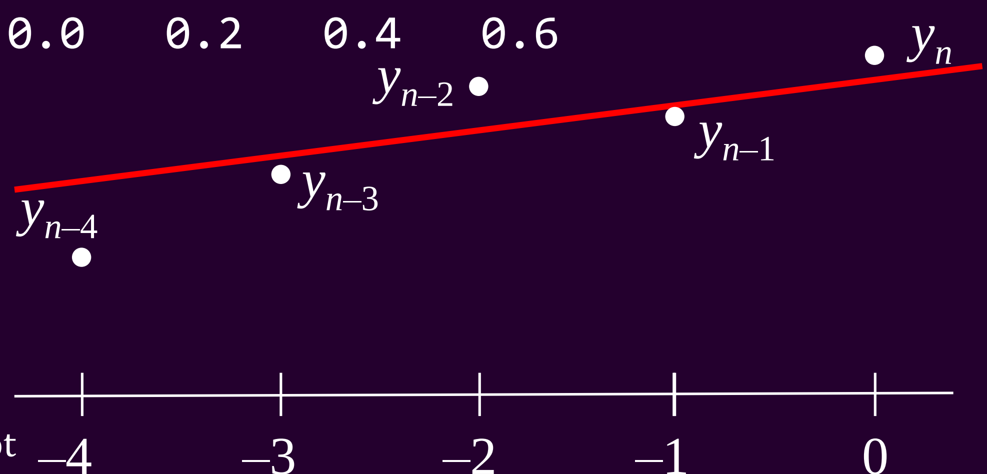


Do not memorize these square matrices or the target vector
- Understand they are the result of calculating A^TA and $A^T\mathbf{y}$

Equally spaced samples

- Fortunately, recall that data tends to be read periodically
  - Let us use the previous practice of shifting and scaling

```
>> A = vander( -4:0, 2 );
>> cond( A )
    ans = 4.738720018687270
>> inv( A'*A )*A'
    ans =
          -0.2   -0.1   0      0.1   0.2
          -0.2    0.0   0.2    0.4   0.6
```



This Matlab code is provided for demonstration purposes and is not required for the examination.

Equally spaced samples

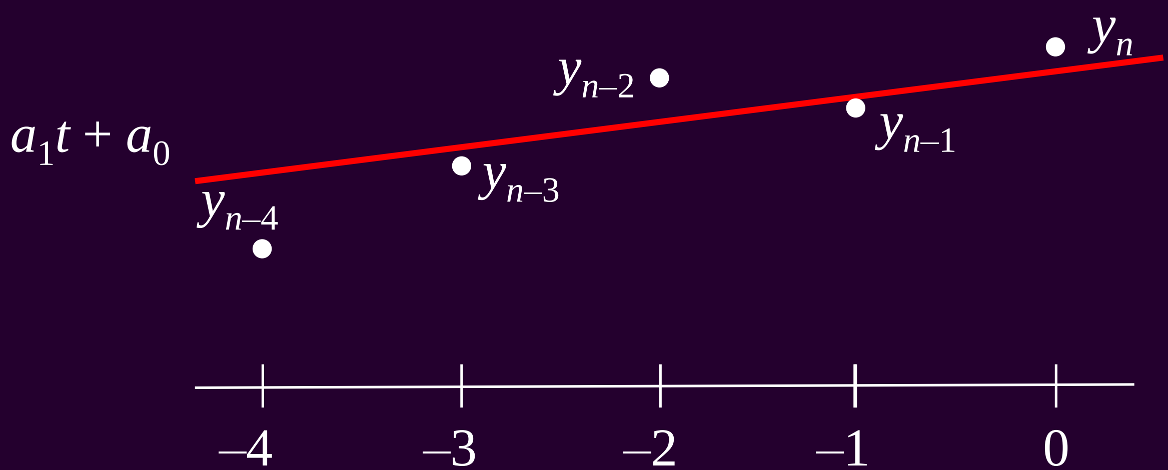
- Thus, we have that

$$\begin{pmatrix} a_1 \\ a_0 \end{pmatrix} = \left(A^{\mathrm{T}}A\right)^{-1} A^{\mathrm{T}}\mathbf{y} = \begin{pmatrix} -0.2 & -0.1 & 0 & 0.1 & 0.2 \\ -0.2 & 0 & 0.2 & 0.4 & 0.6 \end{pmatrix} \begin{pmatrix} y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix}$$

  - More simply, we have that

$$a_1 = -0.2y_{n-4} - 0.1y_{n-3} + 0.1y_{n-1} + 0.2y_n$$

$$a_0 = -0.2y_{n-4} + 0.2y_{n-2} + 0.4y_{n-1} + 0.6y_n$$

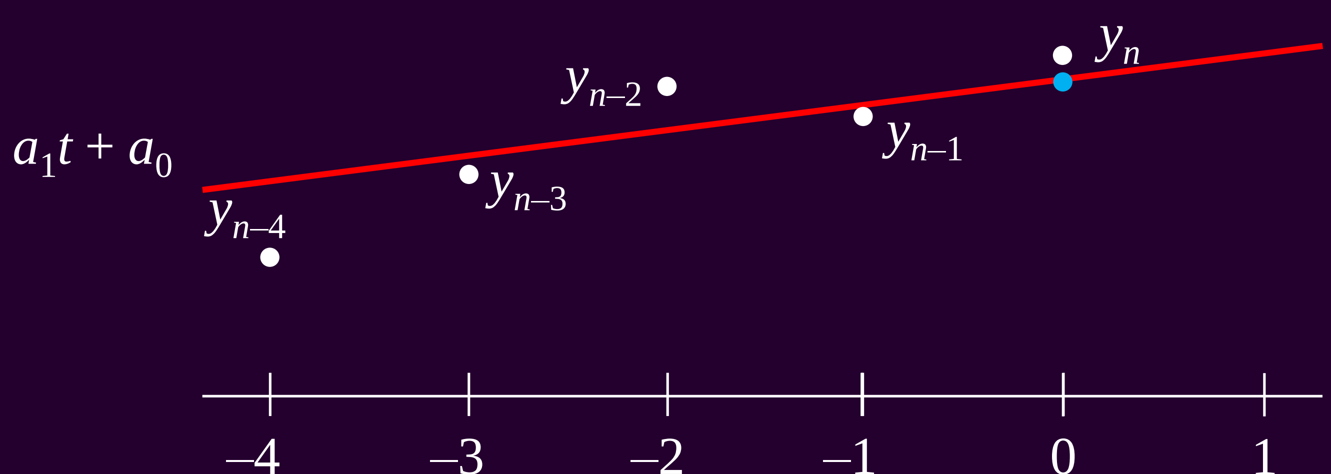


Equally spaced samples

- If the data is noisy, y_n is not even a good approximation of the current value $y(t_n)$
  - Instead, evaluate the least-squares linear polynomial at $t = 0$

    $y(t_n)$ is best approximated by a_0

$$-0.2y_{n-4} + 0.2y_{n-2} + 0.4y_{n-1} + 0.6y_n$$



Equally spaced samples

- We can also estimate the value in the future or around t_n
  - Extrapolate one step into the future by evaluating the least-squares linear polynomial at $t = 1$

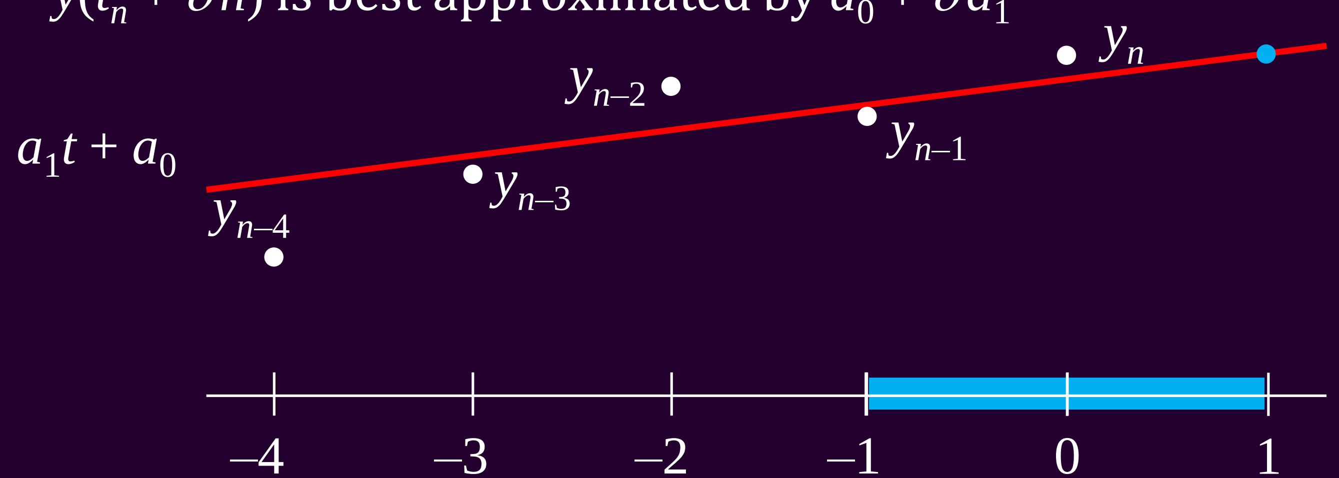
    $y(t_n + h)$ is best approximated by $a_0 + a_1$

    $$-0.4y_{n-4} - 0.1y_{n-3} + 0.2y_{n-2} + 0.5y_{n-1} + 0.8y_n$$

  - More generally, we can estimate the value at $t_n + \delta h$ by evaluating the least-squares linear polynomial at $t = \delta$

    $y(t_n + \delta h)$ is best approximated by $a_0 + \delta a_1$

$$\delta \leftarrow \frac{t - t_n}{h}$$



Equally spaced samples

- Our example uses five points
  - We could choose fewer or more points to find a least-squares line
  - In all cases, a_0 and a_1 are linear combinations of the y values

```
>> A = vander( -9:0, 2 );   # Ten points
>> inv( A'*A )*A'
   ans =
     -0.054545 -0.042424 -0.030303 -0.018182  -0.0060606 0.0060606 0.018182 0.030303 0.042424 0.054545
     -0.14545  -0.090909 -0.036364  0.018182   0.072727  0.12727   0.18182   0.23636 0.29091  0.34545
```

– Having found a_0 and a_1,
our estimators of $y(t_n)$, $y(t_n + h)$ and $y(t_n + \delta h)$ remain unchanged

This Matlab code is provided for demonstration purposes and is not required for the examination.

Equally spaced samples

- Note that because these are integer matrices, we can use some of the properties

```
>> A = vander( -9:0, 2 );   # Ten points
>> detAtA = round( det( A'*A ) )
    detA = 825
>> round( detAtA*inv( A'*A )*A' )
    ans =
       -45   -35   -25   -15    -5     5    15    25    35    45
      -120   -75   -30    15    60   105   150   195   240   285
>> ans/detAtA
    ans =
     -0.054545 -0.042424 -0.030303 -0.018182  -0.0060606 0.0060606 0.018182 0.030303 0.042424 0.054545
     -0.14545  -0.090909 -0.036364  0.018182   0.072727  0.12727   0.18182   0.23636 0.29091  0.34545
```

This Matlab code is provided for demonstration purposes and is not required for the examination.

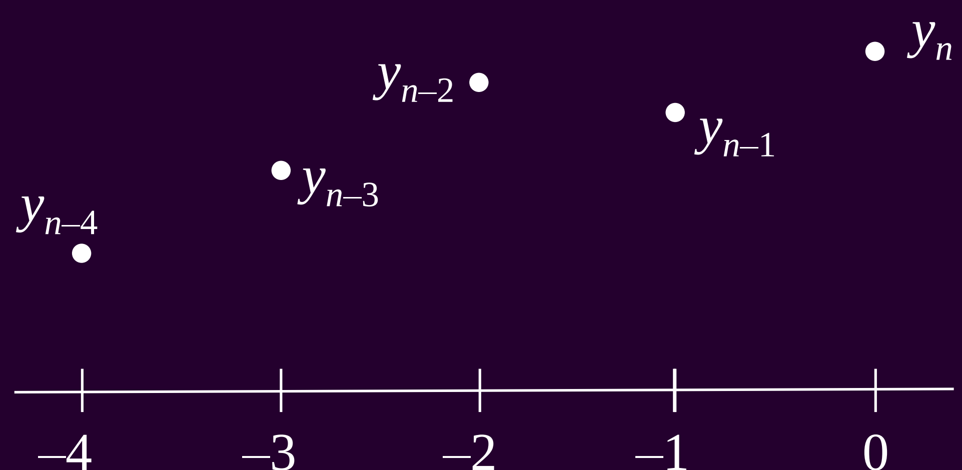
Linear or quadratic least-squares

- Consider this data from a system that is clearly accelerating
  - Using a least-squares linear polynomial would be wrong
  - We should use a least-squares quadratic polynomial

Equally spaced samples

- We can do the same for a least-squares quadratic:

$$\begin{pmatrix} a_2 \\ a_1 \\ a_0 \end{pmatrix} = \left(\left(A^{\mathrm{T}} A \right)^{-1} A^{\mathrm{T}} \right) \mathbf{y}$$



Equally spaced samples

- We can do the same for a least-squares quadratic:

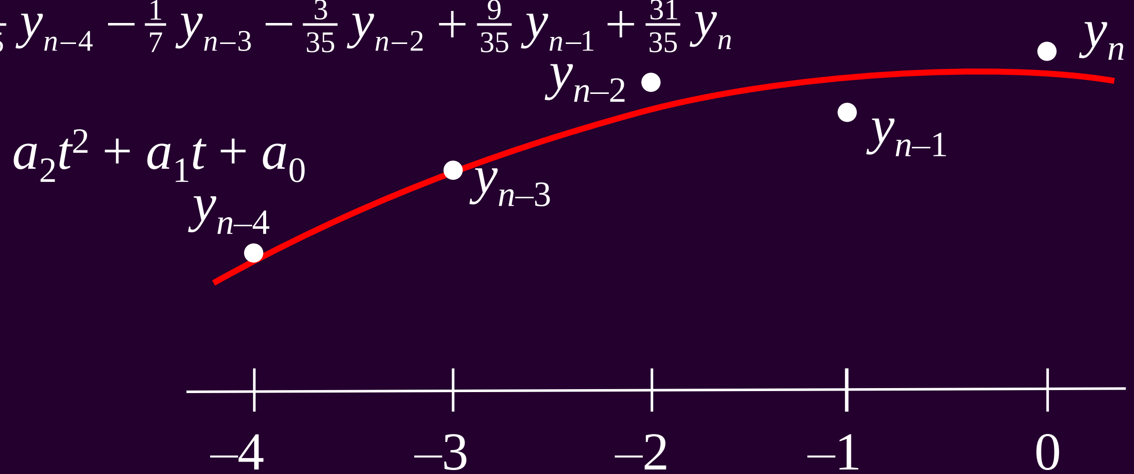
$$\begin{pmatrix} a_2 \\ a_1 \\ a_0 \end{pmatrix} = \frac{1}{\det\left(A^{\mathrm{T}}A\right)}\left(\det\left(A^{\mathrm{T}}A\right)\left(A^{\mathrm{T}}A\right)^{-1}A^{\mathrm{T}}\right)\mathbf{y} = \frac{1}{700}\begin{pmatrix} 100 & -50 & -100 & -50 & 100 \\ 260 & -270 & -400 & -130 & 540 \\ 60 & -100 & -60 & 180 & 620 \end{pmatrix}\begin{pmatrix} y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix}$$

  – More simply, we have that

$$a_2 = \tfrac{1}{7}y_{n-4} - \tfrac{1}{14}y_{n-3} - \tfrac{1}{7}y_{n-2} - \tfrac{1}{14}y_{n-1} + \tfrac{1}{7}y_n$$

$$a_1 = \tfrac{13}{35}y_{n-4} - \tfrac{27}{70}y_{n-3} - \tfrac{4}{7}y_{n-2} - \tfrac{13}{70}y_{n-1} + \tfrac{27}{35}y_n$$

$$a_0 = \tfrac{3}{35}y_{n-4} - \tfrac{1}{7}y_{n-3} - \tfrac{3}{35}y_{n-2} + \tfrac{9}{35}y_{n-1} + \tfrac{31}{35}y_n$$

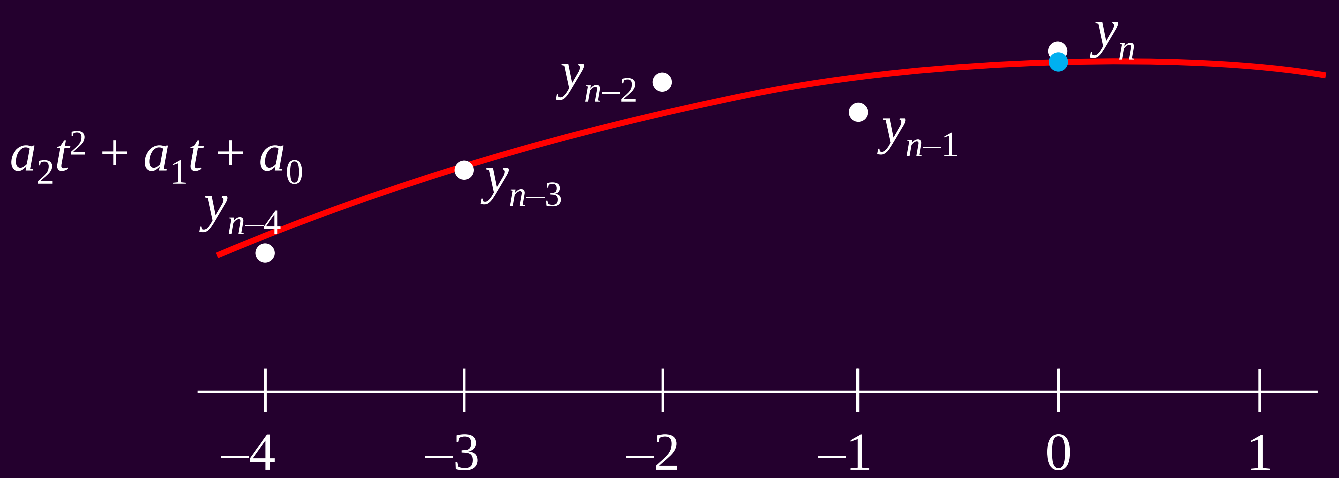


Equally spaced samples

- As before, our best approximation of the actual current value is evaluating this least-squares quadratic at $t = 0$

  $y(t_n)$ is best approximated by a_0

$$\frac{3}{35}y_{n-4} - \frac{1}{7}y_{n-3} - \frac{3}{35}y_{n-2} + \frac{9}{35}y_{n-1} + \frac{31}{35}y_n$$



Equally spaced samples

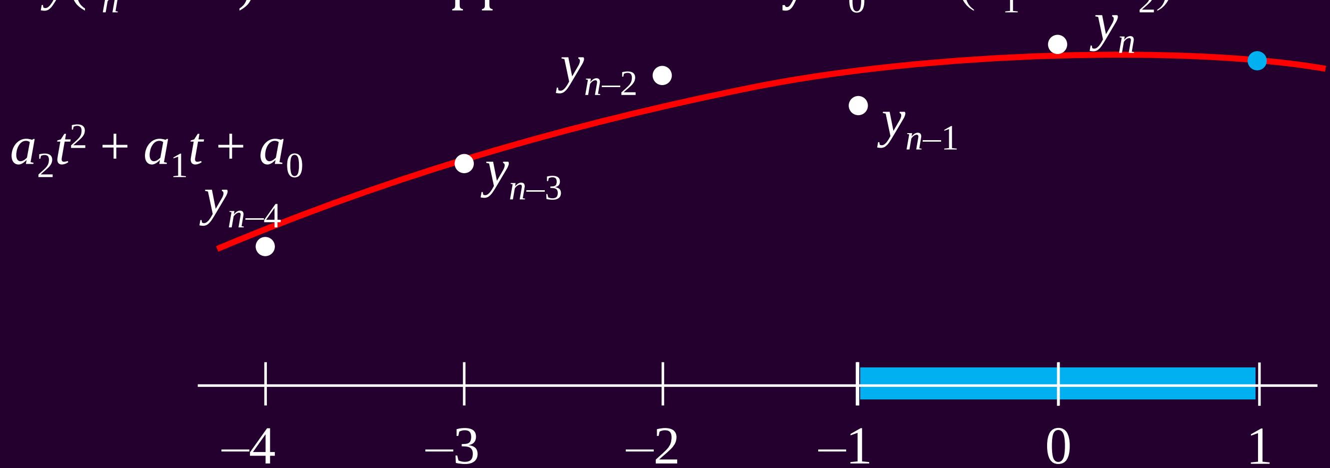
- We can also estimate the value in the future or around t_n
  - Extrapolate one step into the future by evaluating the least-squares quadratic polynomial at $t = 1$

    $y(t_n + h)$ is best approximated by $a_0 + a_1 + a_2$

    $$0.6y_{n-4} - 0.6y_{n-3} - 0.8y_{n-2} + 1.8y_n$$
  - We also estimate the value at $t_n + \delta h$ by evaluating the least-squares quadratic polynomial at $t = \delta$

    $y(t_n + \delta h)$ is best approximated by $a_0 + \delta(a_1 + \delta a_2)$

$$\delta \leftarrow \frac{t - t_n}{h}$$



O(1) run time?

- Issue:
  - This is still a single O(n) calculation with each step
  - You may note that there is a particular pattern

$$a_1 = -0.2y_{n-4} - 0.1y_{n-3} + 0.1y_{n-1} + 0.2y_n$$
$$a_0 = -0.2y_{n-4} + 0.2y_{n-2} + 0.4y_{n-1} + 0.6y_n$$

  - With the next step, the coefficients are now

$$a_1 = -0.2y_{n-3} - 0.1y_{n-2} + 0.1y_n + 0.2y_{n+1}$$
$$a_0 = -0.2y_{n-3} + 0.2y_{n-1} + 0.4y_n + 0.6y_{n+1}$$

  - Let $s \leftarrow y_{n-3} + y_{n-2} + y_{n-1} + y_n$, and so we update

$$a_1 \leftarrow a_1 + 0.2y_{n-4} - 0.1s + 0.2y_{n+1}$$
$$a_0 \leftarrow a_0 + 0.2y_{n-4} - 0.2s + 0.6y_{n+1}$$
$$s \leftarrow s - y_{n-3} + y_{n+1}$$

Summary

- Following this topic, you now
  - Understand that we can easily find formulas for least-squares best-fitting polynomials if the t-values are equally spaced
  - Are aware that with the integer matrices we defined, it is reasonable to calculate $(A^T A)^{-1} A^T$
  - Understand that this allows us to find least-squares best-fitting polynomial coefficients very quickly
  - Know that we can use these coefficients to estimate the value of the function around the current time t_n
  - Are aware that we can even do this constant run time

References

[1] https://en.wikipedia.org/wiki/Least_squares

Acknowledgments

None so far.

Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`. Mathematical equations are prepared in MathType by Design Science, Inc.

Examples may be formulated and checked using Maple by Maplesoft, Inc.

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